Complex Number
Geometrical Applications

1. ABCD is a parallelogram. The angle ABC is α, the ratio of the length of BC to the length AB is $k : 1$, and the sense of description of ABCD is counter clockwise. If A and B represent the complex numbers z_1 and z_2, determine the complex numbers represented by C and D.

2. The complex numbers z_1 and z_2 are connected by the relation $z_1 = z_2 + \dfrac{1}{z_2}$. If the point representing z_2 in the Argand diagram describes a circle of radius a with center at the origin, show that the point representing z_1 describes the ellipse

$$\frac{x^2}{(1+a^2)^2} + \frac{y^2}{(1-a^2)^2} = \frac{1}{a^2} .$$

3. The complex numbers a, b, c, x, y, z are represented in the Argand diagram by points A, B, C, X, Y, Z. If the triangles ABC, XYZ have equal areas, prove that

$$\begin{vmatrix} a & b & c \\ \bar{a} & \bar{b} & \bar{c} \\ 1 & 1 & 1 \end{vmatrix} = \pm \begin{vmatrix} x & y & z \\ \bar{x} & \bar{y} & \bar{z} \\ 1 & 1 & 1 \end{vmatrix} .$$

4. If z_1, z_2, z_3 are complex numbers, interpret geometrically the complex numbers $z_1 - z_2$, $\dfrac{z_1 - z_2}{z_3 - z_2}$.

Triangles BCX, CAY, ABZ are described on the sides of a triangle ABC.
If the points A, B, C, X, Y, Z in the Argand diagram represent the complex numbers a, b, c, x, y, z respectively, and $\dfrac{x-c}{b-c} = \dfrac{y-a}{c-a} = \dfrac{z-b}{a-b}$, show that the triangles BCX, CAY, ABZ are similar.
Prove also that the centroids of ABC, XYZ are coincident.

5. **(i)** If $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{z_1 - z_2}{z_3 - z_2}$, show that the points representing the complex numbers z_1, z_2, z_3 in the Argand diagram form an equilateral triangle.

(ii) The three complex numbers z_1, z_2, z_3 are represented in the Argand diagram by the vertices of a triangle $Z_1 Z_2 Z_3$ taken in counter clockwise order. On the sides of $Z_1 Z_2 Z_3$ are constructed isosceles triangles $Z_2 Z_3 W_1$, $Z_3 Z_1 W_2$, $Z_1 Z_2 W_3$, lying outside $Z_1 Z_2 Z_3$.
The angles at W_1, W_2, W_3 all equal $\dfrac{2\pi}{3}$.
Find the complex numbers represented by W_1, W_2, W_3. Prove that the triangle $W_1W_2W_3$ is equilateral.

6. The cross-ratio of four complex numbers z_1, z_2, z_3, z_4, written (z_1z_2, z_3z_4) is defined by

$$(z_1z_2, z_3z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} .$$

Show that if $(z_1z_2, z_3z_4) = \lambda$, then the 24 permutations of the numbers z_1, z_2, z_3, z_4 give rise to six distinct cross-ratios with values $\lambda, 1 - \lambda, \lambda^{-1}, (1 - \lambda)^{-1}, \lambda(\lambda - 1)^{-1}, \lambda^{-1}(\lambda - 1)$.

If a, b, c, d are four complex numbers, and the complex numbers w_i, z_i $(i = 1, 2, 3, 4)$ are connected by the relation $w_i = \dfrac{az_i + b}{cz_i + d}$ (where $ad - bc \neq 0$), prove that $(z_1z_2, z_3z_4) = (w_1w_2, w_3w_4)$.

If z_1, z_2, z_3, z_4 are distinct and the cross-ratio (z_1z_2, z_3z_4) is real, prove that the points Z_1, Z_2, Z_3, Z_4, representing z_1, z_2, z_3, z_4 lie on a circle.

7. Prove that the transformation $w = \dfrac{2z + i}{2 - iz}$ transforms the unit circle in the z-plane into the unit circle in the w-plane.

8. Let $f(z) = \dfrac{a(z - b)}{1 - z\bar{b}}$, where a and b are complex numbers such that $|a| = 1$ and $|b| < 1$.

Show that **(a)** $|f(z)| = 1$ for $|z| = 1$

and **(b)** $|f(z)| < 1$ for $|z| < 1$.

9. If a complex number $x + iy$ is represented by the matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$, show that multiplication of complex numbers is correctly represented by multiplication of matrices.
How is the relation $i^2 = -1$ represented in terms of matrices ?

Show that the equations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

are equivalent.

Hence, or otherwise, show that, when rectangular axes OX, OY are rotated counter-clockwise through an angle α to become OX', OY', the complex numbers z, z' representing a point of the plane referred to the two sets of axes, satisfy $z' = e^{i\alpha} z$.